Using TOOL-MAKING to contextualise geometry

THE SUPER-SIZE PROTRACTOR

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Engaging students with real-world activities can be effective in supporting the learning of mathematics, "whenever possible, real-world situations will provide a context for both introducing and applying geometric topics" (NCTM 1989). Students saw the activities referred to in this article as part of their real-world, that is, more than examples drawn from a textbook or a world they had not directly experienced.

This article focusses on a student-constructed protractor. The pedagogical discussion is followed by detailed descriptions of the construction steps. These steps together with alternative constructions offer opportunities to explore deeper mathematics. The construction of the protractor requires the use of a range of techniques that provide context for geometry problems involving congruent triangles, properties of equilateral and isosceles triangles and parallel lines. The protractor is a measurement tool used in constructing and measuring angles and is a learning aid for the teaching and learning of trigonometry, similarity and the unit circle. Ways in which the protractor maybe used are listed in Appendix 1.

Context of the project

Formal proof in Euclidean geometry is often seen by high school students as having little connection with their lives. Patricia, the cooperating teacher in the development of these activities said: "I got good grades in high school geometry but I didn't like it and couldn't see the point. Now I have to teach it." Such experiences are inconsistent with seeing mathematics as a social process of sense making (Schonfield, 1992). Constructing the protractor assists students in making connections between the geometry curriculum and the real-world, between branches of mathematics and prior learning. During the construction of the protractor and its subsequent use, students constructed their own mathematical understanding, worked collaboratively and communicated their learning. According to the *Professional Standards for Teaching Mathematics*:

> Educational research offers compelling evidence that students learn mathematics well only when they construct their own mathematical understanding... This happens most readily when students work in groups, engage in discussion, make presentations, and in other ways take charge of their own learning (NCTM, 1991).

The two activities that are described in this article, Super-size Protractor and Unique Triangles provided a shared experience for future class discourse while their hands-on nature added variety and interest to the teaching program. Using the student-constructed protractor as a measurement instrument increased students' sense of ownership and added further meaning and purpose to the construction, that is, the construction project became a part of helping students to see the point.

The activities were developed for a 9th Grade Integrated Mathematics course in a selective High School in New York City. It was the first year of the school; there were limited resources and Patricia had not taught this level of mathematics previously. However, she was enthusiastic about developing her teaching practice. This situation provided an exciting opportunity to support and model engaging hands-on mathematics lessons beginning with students making their own measurement instrument.

Super-size Protractor was implemented as a hands-on activity to provide a context for some geometry problems and for work on proof. During this activity, students used compass and ruler construction techniques to produce a measurement tool for future use.

The protractor was super-sized to improve the accuracy of angle measurement. The standard plastic protractor was not sufficiently accurate for measurements we wanted to make outside the classroom and more sophisticated measuring devices were not an option. Super-sizing the protractor also enables two-figure readings for sine and cosine ratios and space to add other information such as unit circle coordinates thus making the protractor more effective as a learning aid.

Teacher demonstration was used to introduce and supplement the instructions similar to the student handout (see Appendix 2). Students worked in pairs at this stage, and completed the protractor in two 50 minute periods. They were actively involved in the task, appreciating the change in routine and the opportunity for kinesthetic activity. Various constructions (such as the 60° angle construction) raised students' curiosity, and encouraged them, with minimal prompting, to reason why the construction produced an equilateral triangle and hence the 60° angle. Several months later we were using the protractors while introducing the unit circle. A visiting mathematics professional developer discussed the protractor with students and commented upon how well the students were able to articulate and provide reasons for construction steps (see Figure 1).

Figure 1. Students engaged in the construction of the protractor.

Most groups were satisfied with the quality of their protractors, although students' abilities to create a high quality product were impacted by issues with fine motor skills, incorrect use of the tape measure and rushing the construction steps. There were a number of restarts by groups that were not satisfied. One group even chose to begin again in their own time. The finished protractors were then used in the Unique Triangles activity. The Unique Triangles activity (see Appendix 3) was used to encourage a discovery approach to the congruent triangle postulates and theorems. Each group of three students had a different set of constraints, e.g., construct a triangle with sides of 1, 1.5 and 2 metres. They "drew" their triangle on the floor using string for the sides and paper dots for the vertices. It was an activity that created lots of purposeful mathematical discussion. In the second lesson, students measured each side and angle of the other groups' triangles. A class discussion followed with each group describing how they drew their triangle answering the question "Is this the only triangle you can make from the given constraints?" The lesson was designed to facilitate active engagement in the learning process, and comparison and communication with other group's work. The large scale facilitated a high level of cooperative behaviour within each group as students relied on each other to complete various components of the task. Establishing the conditions for congruent triangles followed naturally out of the subsequent class discussion (see Figure 2).

Figure 2. Students using their protractors during the Unique Triangles activity.

Observations

These activities engaged students, catered for multiple learning styles, involved students in constructing their own knowledge and communicating their mathematical understanding. Student feedback indicated that they:

- saw the activities as connected to the real-world:

  > "This was the first time that I have accually [sic] used this kind of math in a real life situation. I feel this is the best way to teach math, to put it in real situations. The way [my teachers] teach is in a way that if you don't get the answer you will understand how to." (Edward)
  >
  > "Math is a lot easier when we can spot it in our everyday lives." (Catherine)

- valued peer collaboration and understood that peer collaboration supported their learning:

  > "We worked in groups, and I think that groups made it easier and everyone helped out and more people is more better plus better understanding." (Cassie)

- developed skill in the use of a measuring instrument by understanding how it was created:

  > "Before building the large protractor I really had trouble using a protractor but after building the large protractor it made it easier for me to see why it worked and how it works. I never thought that I would be doing a project that was fun and that would help me throughout math. I feel more confident about how and why things work after this project." (Michelle)

- were better able to estimate angles and draw realistic diagrams:

  > "This experience was really helpful. From it I am better at simply being able to guess in the realm of an angle measurement." (Samantha)

- appreciated having time and the opportunity to problem solve and construct their own understandings in a supportive environment:

  > "In elementary and Junior high school we sat in class and took notes and did math problems. This [sic] activities were great because I was able to learn by experience and not just my teacher telling me it is right. I had to think and figure out the answers for myself." (Rosie)

- were engaged in purposeful mathematical activity:

  > "I also thought it is a good idea to have projects like making our own tools because it attracts our attention and helps us understand geometry more." (Destiny)

Further, this work provided opportunities to create problems that the students experienced directly making the mathematics more relevant and contextual thus helping students to build connections.

Construction of the protractor: Discussion and mathematics

This section of the article outlines the steps involved in the construction of the Super-size Protractor. As each step is presented, the mathematics of the construction and alternative construction procedures is made explicit and justified along with practical considerations. Developing justifications for these steps was the context for introducing proof.

1. Attach the grid paper to the piece of cardboard.

Graph paper was chosen so the protractor could be used as a unit half-circle with divisions of 0.01. We used 2.5 mm grid paper, printed using the freeware computer program *Graph Paper Printer gpaper.exe*. For backing we used flaps of large cardboard boxes from the school cafeteria with the grid paper aligned along one edge of the board and taped in position; panel board would be a more durable option.

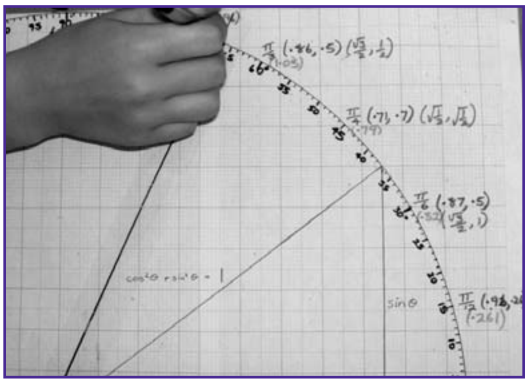

Figure 3. A plastic strip (marked with the black line) is pinned at the origin and being used as a compass. Use of the protractor as a learning aid is facilitated by the use of color and the large size.

2. From the centre draw (as accurately as possible) a semicircle of radius 100.

Students pinned the end of a tape measure at the origin and rotated the tape making numerous marks 10" from the origin. They then drew the semicircle as a freehand curve passing through these points. As an alternative a clear plastic strip, for example a strip from an overhead transparency sheet, could be used as a super-size compass. Pin one end at the origin and make another hole 25 cm away for the pencil or marker to trace out the semicircle as the strip is rotated (see Figure 3).

3. Use the grid to locate the 0°, 90° and 180° points on the semicircle.

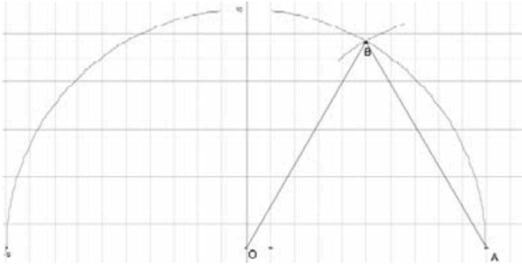

Figure 4

4. Locate 60°.

This is equivalent to swinging an arc of radius 100 units from A intersecting the semicircle at B (see Figure 4). Since OA, OB and AB are equal lengths, ΔOAB is equilateral and $\angle AOB = 60°$.

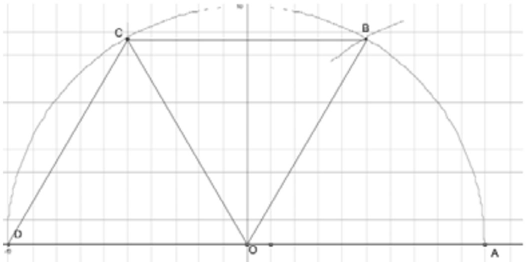

Figure 5

5. Locate 120°.

This is equivalent to swinging an arc of radius 100 units from B to intersect the semicircle at C (see Figure 5). Check that $CD = OA$ (D is 180°). This creates another equilateral triangle OBC with $\angle BOC = 60°$ and $\angle AOC = \angle AOB + \angle BOC = 120°$.

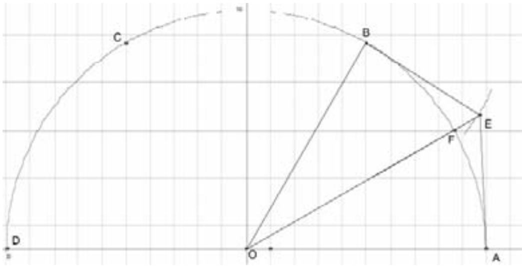

Figure 6

6. Bisect the angle between 0° and 60° to locate 30°.

Use a compass to swing arcs from A and B that intersect at E (see Figure 6). F (the 30° mark) is the intersection of OE and the semicircle. $AE = BE$, $OA = OB$ and OE is common to ΔAOE and ΔBOE. Therefore $\Delta AOE \cong \Delta BOE$, $\angle AOG \cong \angle BOG$ and OG bisects $\angle AOB$. Thus $\angle AOG = \frac{1}{2}(\angle AOB) = 30°$.

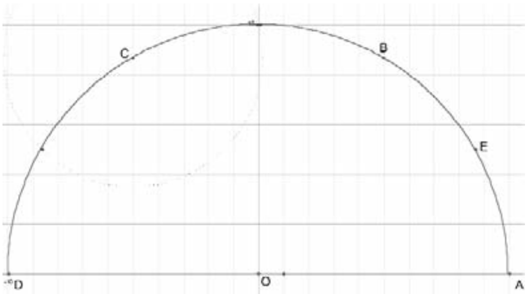

Figure 7

7. Bisect the angle between 120° and 180° to locate 150°.

Alternatively, set a compass to length AE and swing an arc from C (see Figure 7).

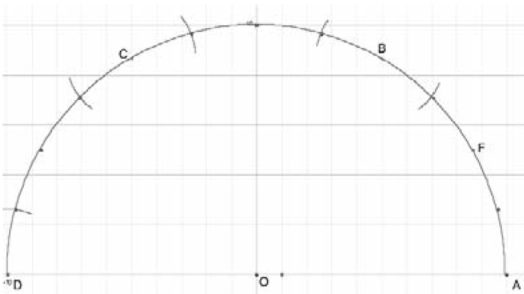

Figure 8

8. Locate all remaining multiples of 15°. Select appropriate angles to bisect

Bisect $\angle AOF$ to locate the 15° mark (see Figure 8). The multiples of 15° can be located by setting the compass to AG and swinging arcs from B, C and D (see Figure 9).

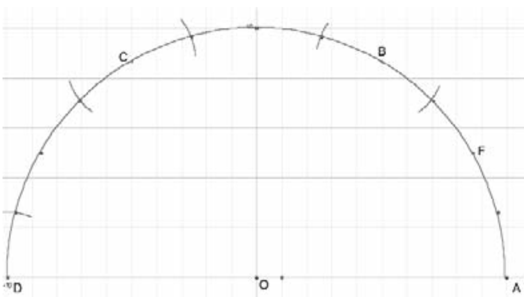

Figure 9

9. Estimate the positions for all multiples of 5° on your semicircle.

Since the aim was to produce a protractor with one degree divisions to be used as a measuring tool, it is necessary to trisect an angle. We chose to do this by estimation. This approach was easily understood by students, required minimal class time and was sufficiently accurate for the proposed uses. Once this was satisfactorily done for one 15° interval, we set a compass to the chord length of the 5° arc and marked off the semicircle in 5° intervals (see Figure 10).

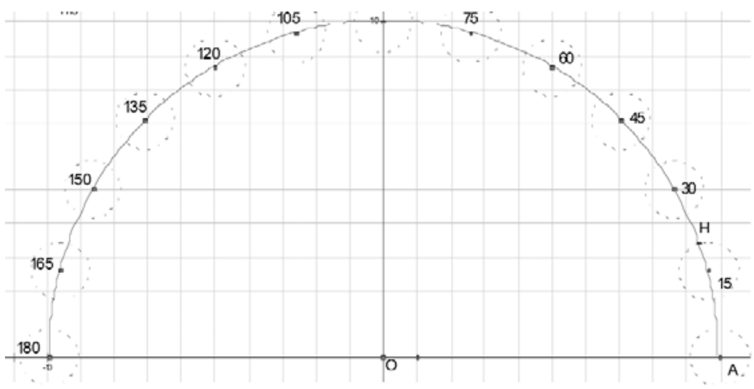

Figure 10

A compass and straight edge construction trisection of a general angle is impossible, proved algebraically by Wantzell in the 19th century (Weisstein, 2003). However, a highly accurate approximation can be obtained by trisecting the chord of a 15° arc. Draw a chord of a 15° arc, measure its length, divide by 3 and measure off this distance. Figure 11 (drawn with interactive symbolic geometry software) shows the m∠*AOL* to be 4.987°, an error of less than 0.013° or 0.0026%.

Archimedes' method uses a marked straight edge and compass to trisect any angle. If Archimedes' method is used to construct a 5° interval on the protractor then trisecting 60° is suggested (see Figure 12).

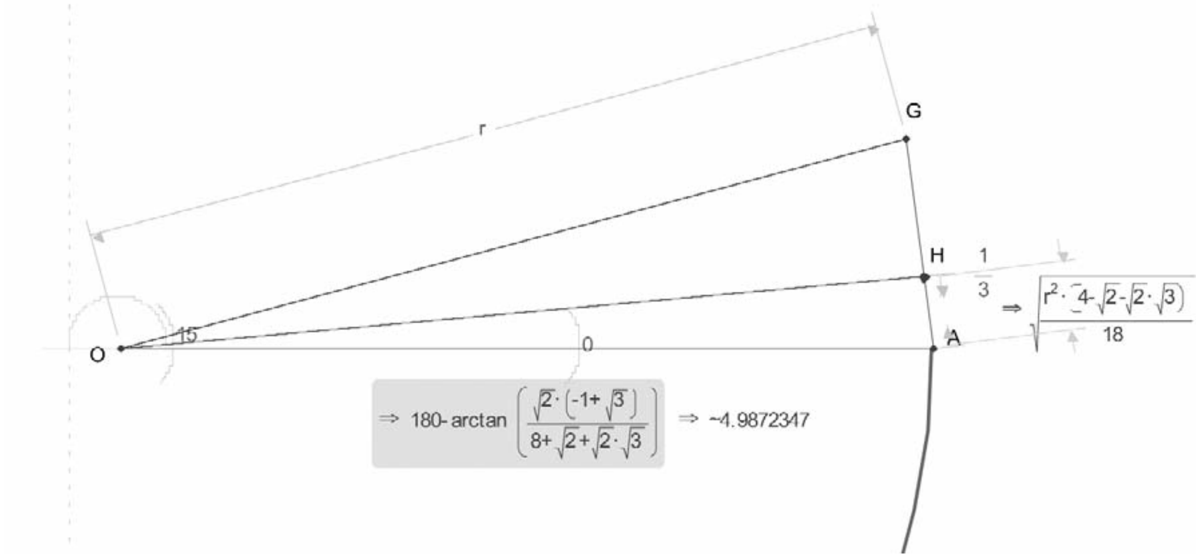

Figure 11

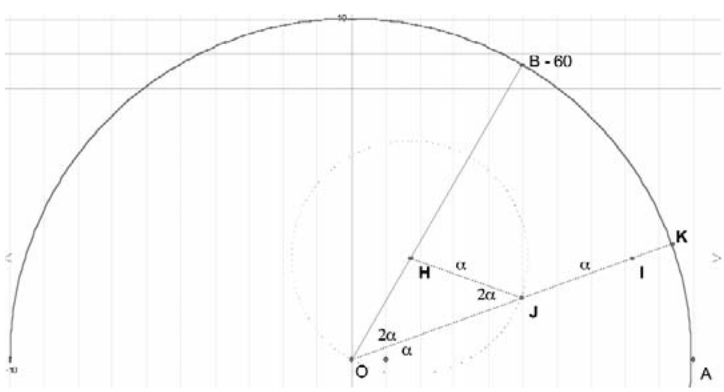

Figure 12

- Place a point H about a third of the way along OB on a horizontal grid line (this distance is suggested so that the construction will fit on the grid paper).
- Construct a line l_1, parallel to OA through H (a horizontal grid line).
- Construct a circle C_1 centre H passing through the origin O.
- Measure the radius of C_1.
- Orient the ruler so that the ruler's zero lies on line l_1, the measured radius of C_1 lies on the circle and the ruler passes through O. Mark the intersection of the ruler with C_1. Extend this line to the semicircle. The intersection K is the 20° mark.

Proof

Let $m\angle AOK = \alpha$	
$m\angle HIO = \alpha$	alternate interior angles
$HJ = IJ$	construction property
ΔHIJ is isosceles	2 equal sides
$m\angle JHI = m\angle JIH = \alpha$	base angles of isosceles triangle
$m\angle OJH = m\angle JHI + m\angle HIJ = 2\alpha$	Exterior angle to a triangle
$HJ = OH$	radii of C_1
ΔOHJ is isosceles	2 equal sides
$m\angle HOJ = m\angle HJE$	base angles of isosceles triangle
$m\angle HOA = 60°$	given

$$\begin{aligned} m\angle HOA &= m\angle HOJ + m\angle JOA \\ &= 2\alpha + \alpha \\ &= 3\alpha = 60° \end{aligned}$$

$\alpha = 20°$

10. Estimate one degree intervals and mark these.

Estimation was quick and sufficiently accurate. The quinsection of the chord of a 5° arc creates central angles within 0.001° of a degree should a construction method be preferred to estimation.

Conclusion

Tool making is a way to bring real-world mathematics into the classroom and using the tool is a way of taking classroom mathematics into the student's real-world. I met Paul, a student from this class outside school and he commented "My other teachers [Mathematics] are interested in talking about math in the real-world. You wanted to get us out into the real-world."

Activities based around the construction of the super-size protractor were effective in engaging students in building and using their own measurement tool, supporting collaborative learning, constructing their own mathematical knowledge and creating a context for geometry proof. Connections were implicit in the protractor's use in many topics of the course and, most significantly, students saw this mathematics as part of their real-world.

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Appendix 1: Protractor uses

The protractors can be used:

- to provide problems requiring justification or proof;
- to measure angles;
- as inclinometers;
- to draw scale diagrams of right triangles. Measurement and similarity arguments are then invoked to solve the triangles. (The students demonstrated little conceptual understanding of right triangle trigonometry despite showing procedural fluency in the use of SOHC-AHTOA);
- as a unit half-circle. Having the protractor superimposed on a 0.01 unit grid enables coordinates to be read off easily;
- as a learning aid in the memorization of the coordinates of points representing multiples of 30° and 45° on the unit circle;
- as a learning aid in the memorization of particular sine and cosine ratios;
- to teach radian measure of angles. Simple fractions of π can be written next to the degree measure to assist with retention. Using different colours for degree measure, radian measure, coordinates etc. enhances the protractor's use as a learning aid;
- to re-conceptualize sine and cosine as functions; and
- to use symmetry arguments to develop and justify trigonometric identities. Students noted the symmetry about the y-axis in the construction process, i.e., $\sin(\pi - x) = \sin x$.

Appendix 2: Super-size protractor

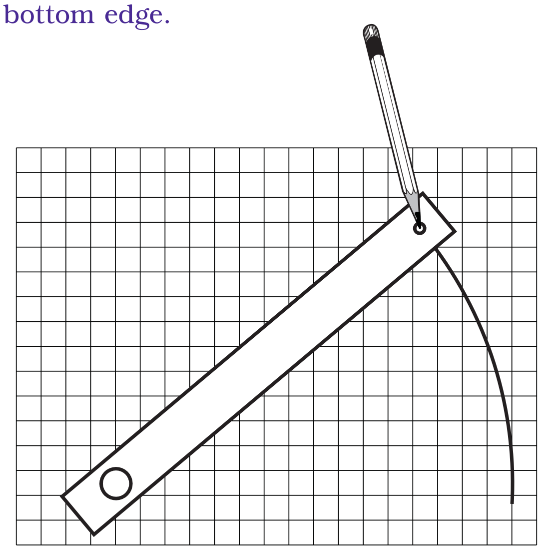
Materials required (per group)

- Piece of board or cardboard backing: at least 50 cm x 25 cm
- 3 sheets of 2.5 mm graph paper
- Scissors
- Tape
- Strip of overhead transparency and drawing pins or compass able to draw 30 cm radius arc
- Ruler

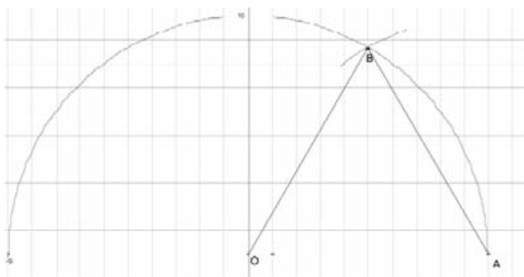
Instructions

We will be using the protractors you make for measuring angles later in the course. So it is important to make the constructions as accurately as you can. Do each step in pencil and when you are happy that the line is in the right position go over the lines with a fine point permanent marker.

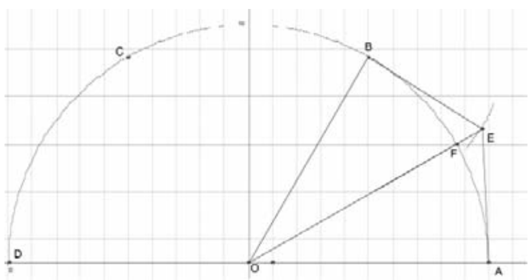
1. Setting up.
   - Trim off the edges of the grid paper.
   - Tape the grid paper to the piece of cardboard, taking care to line up the grids (it should appear as a single piece of grid paper at least 200 by 100), and aligning with the bottom edge of your cardboard. Mark the centre in the middle of the bottom edge.
2. From the centre draw (as accurately as possible) a semicircle of radius 100.
   - Make a hole in one end of your transparency strip.
   - Pin the strip to the centre of the bottom edge.
   - Make a second hole at the 100 unit mark, just large enough to put your pencil or marker through.
   - Rotate the strip to draw your semi-circle. The semicircle should pass exactly through the 100 point of the grid paper at the bottom on both ends and the top.



3. Use the grid to locate the 0°, 90° and 180° points on the semicircle.
4. Locate 60°.
   - Pin your strip at the 100 mark on the bottom (Point *A* in the diagram).
   - Swing an arc across the semicircle.
   - The intersection is the 60° mark.



5. Locate 120°.
   - Pin your strip at the 60° mark. Swing an arc across the semicircle. The intersection is the 120° mark.
6. Bisect the angle between 0° and 60° to locate 30°.
   - Make a second hole in your strip about 70% of the semicircle radius.
   - Pin your strip at 0 and swing an arc.
   - Pin your strip at 60° and swing an arc.
   - Place your ruler between the centre of the grid and the intersection of the arcs.
   - Mark the intersection of the semi-circle and the ruler.



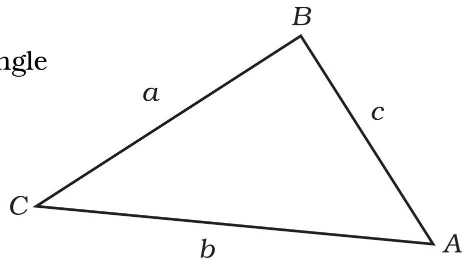
7. Bisect the angle between 120° and 180° to locate 150°.
8. Locate all remaining multiples of 15°. Select appropriate angles to bisect.

9. Estimate the positions for all multiples of 5° on your semicircle (or use a construction method like that shown at www.geom.uiuc.edu/docs/forum/angtri).
10. Estimate one degree intervals and mark these in as well.

Appendix 3: Unique triangles

Materials

- Super-size protractors
- Stick on dots to mark vertices of the triangle
- String or wool to represent the sides of the triangle
- Measuring tapes



Procedure

This is a group activity.
Note that the side of length a is opposite vertex A.

1. Your group will be given some properties of one triangle to construct. Plan how you will construct your triangle.
2. On the floor construct a triangle with the properties given.
3. Measure the missing sides and/or angles of your triangle.
4. Talk about your triangle with your group members and practice how you will demonstrate the construction of your triangle to the rest of the class.
5. When all groups have finished, measure all sides and angles of the other triangles to check and complete the table.

	a (m)	B (m)	c (m)	A	B	C	Is the triangle unique?
a)	2	1	1.5				
b)				45°	80°	55°	
c)	4 m	2’	1				
d)		2	3	55°			
e)		2		60°	45°		
f)			1	120°	25°		
g)	2.5	2		90°			
h)	2.5		2			45°	
i)		1.5		73°	49°		

6. Predict which descriptions will produce a unique triangle.

From Helen Prochazka's Scrapbook

Barack Obama in his book “The Audacity of Hope” (Random House, 2006) on page 22 writes:

We know that global opportunity — not to mention any genuine commitment to the values of equal opportunity and upward mobility — requires us to revamp our educational system from top to bottom, replenish our teaching corps, buckle down on math and science instruction, and rescue inner-city kids from illiteracy.